Introduction to Markov Chains

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- Markov chains are named after Russian mathematician Andrei Markov (1856-1922).
- The Markov chain is a model for an experiment consisting of a sequence of trials for which the outcome of each trial depends on the outcome of the previous trial.

States and Transition Matrices

- A *Markov chain* is given by a set of states and a transition matrix
- In many applications we have a set of *states* $\{S_1, S_2, ..., S_n\}$
- Example: voters may vote Democratic, Republican or for a third party.
- The probability that a member of the population will change from the ith state to the jth state, the *transition probability* is given by $p_{i,j}$ where $0 \leq p_{i,j} \leq 1$.
- In particular, $p_{i,i}$ gives the probability that a member of the population will remain in the ith state. A probability $p_{i,j} = 0$ means that it is certain not to change state; $p_{i,j} = 1$ means that the member is certain to change state from S_i to S_j.

States and Transition Matrices

- All these probabilities form the *Transition Matrix*

$$P = \{p_{i,j}\}, 1 \leq i \leq n, 1 \leq j \leq n$$

- At each transition, each member in a given state either transitions to another state or stays in the same state.
- Since the entries in the jth column represent the probabilities that a member of the population will change to the jth state, the sum of the transition probabilities is 1, i.e.,

$$p_{1,i} + p_{2,i} + ... + p_{n,i} = 1$$

Stochastic Matrix

- A $n \times n$ matrix P is called *stochastic*, if each entry in the matrix is a number is a number between 0 and 1 and each column of P adds up to 1.

$$0 \leq p_{i,j} \leq 1$$

$$\sum_{i=1}^{n} p_{i,j}, 1 \leq j \leq n$$

The *m*th State of the Markov Chain

- $P^m = P.P...P$, where the matrix P is multiplied with itself m times
- The mth state of the Markov chain with P as the transition probability matrix and X_0 as the initial state vector is

$$X_0 = P^m X_0$$

Regular Markov Chains

- We are now concerned with the *long-term* behavior of the Markov chain after many trials.
- A stochastic matrix P is called *regular* if some power of P has only positive entries.

Theorem

- *If P is a regular stochastic matrix, then the sequence*

$$P, P^2, P^3, P^4 \ldots$$

approaches a stationary matrix $\bar{P}$.

- *For a vector X, the corresponding Markov chain is called* aregular Markov chain *and the sequence:*

$$PX, P^2X, P^3X, P^4X \ldots$$

approaches $\bar{X}$ which we call the stationary distribution vector.

- *Furthermore, the entries in each column of $\bar{P}$ are equal to the corresponding entries in the stationary distribution vector $\bar{X}$. That is, all columns of $\bar{P}$ are the same, all equal to $\bar{X}$.*

Algorithm for Finding the Stationary Distribution for Regular Markov Chains

Theorem

- *Solve the system of linear equations*

$$PX = X$$

$$x_1 + x_2 + ... x_n = 1$$

where

$$X = (x_1, x_2, ..., x_n)$$

- *the solution*

$$\bar{X} = (\bar{x_1}, \bar{x_2}, ..., \bar{x_n})$$

is the stationary distribution of the Markov chain.

- *the stationary matrix is*

$$\bar{P} = (\bar{X}, \bar{X}, ..., \bar{X})$$